A New Model for Mixing and Fluctuations in a Turbulent Wake

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A new model for turbulent wake mixing is proposed. The entrainment of surrounding fluid by the growth of the turbulent core in a high-speed wake is pictured as introducing into the core fluid elements that retain for some time properties characteristic of their state prior to entrainment. This leads to a two-component wake structure, consisting of well mixed fluid on the one hand, and unassimilated fluid on the other. The delay in complete assimilation is represented in the model by a mixing lag. A straightforward set of equations describing the proposed (one-dimensional) model is derived. The two-component model leads to a prediction of mean-mass and electron-density fluctuations in the wake. The fluctuations are assumed to arise from random convection of inhomogeneous fluid elements, rather than by compressibility effects or dissipation associated with the turbulent velocity field, which are shown to be small by comparison. Simple approximate expressions for the relative-mass and electron-density fluctuations are derived which show that the two are not generally equal, as has often been assumed in computations of radar backscatter from wakes. Numerical predictions for the magnitude and variation of mass density fluctuations in a relatively low-speed wake are obtained by solving a simplified set of equations, in which the lag is represented by an effective mixing boundary, and are in encouraging agreement with the general magnitude and trend observed experimentally, although firm experimental measurements are lacking at this time.

1. Introduction

THE passage of a hypervelocity object through the atmosphere generates, at sufficiently high Reynolds numbers, a wake consisting of two distinct portions. The inner (viscous) portion is turbulent and the outer (inviscid) portion is laminar. The inner wake grows with downstream distance to engulf the outer flow, and the wake turbulence plays a major role in the wake growth, and in the mixing of the outer flow into the turbulent core. In order to predict the radar backscatter from such turbulent wakes, it is necessary to know some of the statistical properties of the dielectric constant field, and hence, of the electron-density field of the wake as a turbulent medium. In particular, the mean intensity of the electron-density fluctuations must be known. Direct measurements of the electron-density fluctuations in a turbulent wake plasma do not exist in the literature at this time. Rothman et al.,[1] have performed such measurements for the case of a rocket exhaust, using an ion probe and a microwave electron probe. In the case of wakes, however, only indirect and spatially unresolved measurements such as radar backscatter measurements[2] have been made. On the other hand, laboratory measurements[3, 4] of the fluctuations of mass density in the wake of hypersonic projectiles using shadow photographs and densitometry techniques can determine the latter with good spatial resolution (though their interpretation is not entirely unambiguous). Thus, it would be very useful to be able to relate the electron- and mass-density fluctuations in the wake. In fact, the amplitudes of electron- and mass-density fluctuations have been roughly equated in computations of wake radar cross sections. As will be shown, such a procedure is not generally justified. In the present paper, the mean-electron and mass-density fluctuations in turbulent wakes are estimated on the assumption that they arise mainly through the mixing of outer wake fluid into the turbulent core, by random convection of engulfed fluid elements into the volume of the wake core. The mixing model used is based on a simple phenomenological description of gas mixing in a turbulent wake, which is consistent with our current understanding[5] of low-speed turbulent wake growth and mixing, as well as with experimental evidence of large amplitudes of the mass-density fluctuations obtained in laboratory measurements.[3, 4, 6] Fluctuations arising from compressibility effects, or from temperature fluctuations associated with the turbulent velocity field are argued to be small by comparison, and too small to explain the observed magnitude of the fluctuations.

Although the interest in hypersonic turbulent wakes is centered on the case of high-speed wakes, mass-density fluctuation measurements have so far been confined to relatively low velocities. Thus, the most striking effect of the present model, which is its influence on the wake chemistry, and particularly the ionization chemistry, cannot be directly tested at this time. However, the ability of the model to reproduce the gross features of the mass-density fluctuations at the lower speeds is an encouraging indication that it may be applied with some confidence to describe the properties of hypersonic re-entry wakes.

2. Turbulent Wake Mixing and Wake Structure

Theoretical descriptions[7] of the turbulent hypersonic wake have so far been largely based on an extension of laminar hypersonic wake analyses, essentially by the introduction of an eddy transport coefficient to represent the effects of turbulent convection and dissipation. While this approach appears to predict such gross features as wake growth rather well, it cannot inquire into the physical problem of turbulent mixing, since it avoids the problem a priori by returning effectively to a quasi-laminar wake. On the other hand, the chemistry of the turbulent wake is strongly dependent on the nature of that mixing. In the present approach, an attempt has been made to retain the characteristic feature of turbulent mixing, namely, the combined action of random convection on the one hand, and molecular diffusion aided by the distorting effects of turbulent motion on fluid elements, on the other. This is

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done at the expense, at least for the time being, of any sophistication in the fluid dynamical formulation of the problem.

2.1 Wake Growth and Mixing

The wake of hypersonic blunt bodies consists,[7] at sufficiently high Reynolds numbers, of an inviscid outer wake and a viscous turbulent inner wake. This wake structure is well known and need not be described here. It is sufficient to recall that the turbulent wake consists initially (i.e., at the neck) of the gas contained in the free shear layers shed from the body upon separation of the boundary layer and that it grows with downstream distance, and eventually engulfs the entire (originally inviscid) outer wake.

Experimental observations of turbulent wakes in hypervelocity ranges[3, 4] and of turbulent subsonic wakes[8, 9] reveal that the turbulent wake possesses a sharp front which separates it from the nonturbulent outside fluid. According to presently accepted notions in the theory of turbulence,[5, 8] the turbulent wake consists of eddies of widely varied sizes. Most of the kinetic energy of turbulence is contained in the relatively larger eddies, whereas the smaller eddies are responsible for most of the viscous dissipation of kinetic energy into heat. The very largest eddies do not contain much energy, but they are important in contorting the wake surface, at least in subsonic wakes. The energy containing eddies have much shorter lifetimes. In general, the eddy lifetime decreases rapidly with eddy size.

The growth of the turbulent wake and the manner in which it engulfs outer fluid may be broadly characterized as follows: the local propagation of the turbulent front bounding the wake core occurs by diffusion of vorticity fluctuations into the outer fluid.[8, 9] The diffusion is initiated by the smallest eddies, since they represent the greatest velocity gradients. The transfer of turbulence energy to the previously nonturbulent fluid is accomplished by the energy containing eddies, which transfer energy to the vorticity diffused into the outer fluid by the small eddies. In the preceding manner, turbulence is generated into previously quiescent fluid. The turbulent wake "front" represents a boundary between fluid which possesses vorticity and vorticity fluctuations, and fluid which does not. Corrsin and Kistler[9] describe the front as a "laminar superlayer," of very small thickness, across which vorticity decreases from values characteristic of the turbulent flow, to very low values. Vorticity is communicated locally to new fluid across this thin layer, principally by the smallest eddies of the turbulent flow, resulting in a propagation of the front into new fluid. The new fluid thus acquires vorticity, and vorticity fluctuations, but it acquires no large scale random motions by the preceding mechanism. Consequently, the newly engulfed fluid would tend to remain close to its nonturbulent trajectory, under the effect of the primary turbulence propagation mechanism. It would, therefore, not be really mixed as yet, and the small scale motions induced in it would not appreciably aid the molecular mixing process.

If there were no other mixing mechanism, each additional layer of engulfed fluid would come in contact with only the preceding layer of engulfed fluid, and effective mixing would not occur. Effective mixing would require a mechanism which swept away newly engulfed fluid and mixed into the "deeper" parts of the flow, while exposing new layers to fluid brought up from those deeper parts. Such a mechanism does indeed exist, but it is quite distinct from the primary entrainment mechanism, and is due to the turbulent motions on another scale. Indeed, in addition to the small scale eddies, the turbulent flow exhibits larger scale eddy motions, which convert turbulent fluid from one part of the wake to another. (The large eddies also serve to contort the turbulent front and, thereby, greatly increase the rate of entrainment of new fluid, since the latter depends on the total area of the "laminar superlayer.") The newly engulfed fluid will not become mixed into the core until it is convected by one of those larger eddy motions. The scale of the inhomogeneities of density, temperature, etc., introduced into the turbulent core in this manner, depends then on how much new fluid is engulfed by the propagation of the turbulent front. It may be roughly estimated as follows. The main convective action of the turbulence is carried on by the larger eddies of the turbulent flow, of scale L, say. The time required by those eddies to sweep away newly engulfed fluid is of the order $\tau \sim L/u'$, where u' is the mean velocity of the larger eddies. Over a period of time τ the turbulent front will propagate a distance $D \sim V^*\tau$, where V^* is the mean propagation velocity of the front. The "transverse" scale of the engulfed fluid will be of the order of D, while its "longitudinal" scale (along the boundary) will be some fraction of L. These scales may now be compared with the scales of inhomogeneities at which molecular diffusion becomes dominant. The latter scale is comparable to the dissipation or Kolmogorov scale when the kinematic viscosity and diffusivity are comparable. In a well developed turbulent flow, it can be expected to be two or more orders of magnitude below the macroscale of turbulence, namely L. The question remains whether D is also larger than the Kolmogorov scale. From the foregoing discussions, we may write

$$D \sim V^*\tau \sim L\frac{V^*}{u'} = L\left(\frac{V^*}{V_\infty}\right)\left(\frac{V_\infty}{\Delta V}\right)\left(\frac{\Delta V}{u'}\right) \qquad (1)$$

where V_∞ is the freestream velocity, and ΔV is the velocity difference across the wake. The ratio $(V_\infty/\Delta V)$ of the freestream velocity to the wake velocity defect can be roughly taken as[11] $(V_\infty/\Delta V) \sim 8y_f^2\,(\rho/\rho_\infty)$, where y_f is the nondimensional wake radius, normalized with respect to the body diameter, and (ρ/ρ_∞) is the ratio of mean wake to freestream density. The ratio $(\Delta V/u')$ is approximately 2, according to subsonic turbulent wake results. The ratio (V^*/V_∞) is approximately the same as the wake "slope" (d_{y_f}/d_x). Assuming an approximate (nondimensional) wake growth law of the form $y_f \sim 0.5x^{1/3}$, where x is downstream distance expressed in body diameters, the foregoing estimates can be substituted in Eq. (1), which yields

$$D = \tfrac{2}{3}L(\rho/\rho_\infty) \qquad (2)$$

In the region beyond the first ten or twenty body diameters where the present model can be expected to apply, the mean density ρ in the wake will not be less than a few times below the freestream density, so that D will be of the order of $\frac{1}{10}$ of L, which may reasonably be assumed to be significantly greater than the Kolmogorov scale. Thus it appears reasonable to assume that the combined effects of the propagation of the turbulent front and the convective action of the large eddies of the turbulence tend to introduce into the wake fluid elements of a scale large compared to the Kolmogorov or dissipative scale of the turbulence, so that these must be reduced in scale by the cascading process of turbulence before effective molecular mixing can occur.

In summary, the engulfment of the outer wake by the turbulent core may be pictured as a process whereby relatively large fluid elements (i.e., large compared to the Kolmogorov scale) are convected into the turbulent core by the action of large scale eddies, while at the same time, they are distorted by the shearing effects of small scale eddy motions[10] until the scale of the inhomogeneities has been reduced (and the gradients steepened) to the point where molecular diffusion erases those inhomogeneities and mixes the engulfed fluid with the core fluid on a molecular scale.‡

‡ In the absence of molecular diffusion and conduction, the effect of eddy diffusivity (or random convection) is to break up large volume elements into smaller ones (down to the cutoff scale) and, therefore, to increase mean gradients rather than decrease them. The increase in mean gradients leads, however, to increased molecular dissipation, so that eddy diffusion indirectly leads to dissipation of inhomogeneities.

2.2 Physical Model of Wake Structure

As a result of the mixing process described in the preceding section, the wake can be expected to consist essentially of two distinguishable components, namely, a component that has not been mixed on a molecular scale with the main flow, and whose properties are close to the properties it would have if it were not engulfed in the core, and a more or less homogeneous component which consists of the initial core fluid augmented by that portion of the engulfed fluid which has been molecularly mixed. Since both the entrainment process and the mixing process are random in nature, the foregoing description of the wake must be assumed to apply in the mean, so to speak.

The model for the wake structure formulated below assumes that there is no significant fluid component, which is neither unmixed nor completely (molecularly) mixed. This assumption is more important when chemical reactions are concerned, in which case the times characteristic of chemical reactions must be long compared to the time required for fluid elements to "jump" across the Kolmogorov scale. However, the basic concept used in the model, namely, that a macroscopic engulfment and mixing precedes molecular mixing, is not dependent upon that assumption.

In the model of wake structure previously described, the spatial variations in wake properties, such as mass density, electron density, and temperature, are regarded as arising from the existence of "unmixed" fluid elements intermingled with "mixed" fluid, as well as, from inhomogeneities which may exist within or between unmixed fluid elements. Inhomogeneities in the mixed component are assumed small, and will, in fact, be neglected in the following.

The mean-density fluctuations that can be predicted on the basis of a two-component wake model such as the one described previously are in semiquantitative agreement, as will be shown below, with the large mass-density fluctuations apparently found [3–5] in schlieren measurements of density fluctuations in the wakes of hypervelocity pellets at large downstream distances from the pellets. The apparent magnitude of the measured fluctuations at large downstream distances, of the order of one to several tenths at an axial distance of several hundred body diameters, would seem to rule out the possibility that the fluctuations are created within the turbulent flow by compressibility and dissipative effects. Indeed, it is shown in the Appendix that such effects would lead to considerably smaller fluctuations. Similarly, interaction between the turbulence and the wake chemistry must be ruled out for the relatively low temperatures in the observed ballistic range wakes.

The net effect of the mixing process discussed previously is to eventually mix any fluid element entering the turbulent core homogeneously with the core gas. The mixing does not occur instantaneously as the fluid is entrained inside the turbulent front, however, but rather after some lag in time, which is required for the turbulent "shredding" and molecular diffusion to take effect. This lag time and corresponding lag distance over which the fluid element is transported by the main flow velocity will depend on the intensity of turbulence and on the viscosity and diffusivity in the wake. The foregoing mixing model is therefore intermediate between the homogeneous mixing model, which assumes instantaneous mixing, and the inviscid random convection model of Corrsin,[11] in which it is assumed that molecular mixing does not occur at all. The two preceding extremes correspond to a zero and infinite lag, respectively. A quasi-one-dimensional model for (chemically reacting) turbulent wakes has been analyzed by Lin and Hayes[12] for the extreme cases of homogeneous-mixing and inviscid-random convection. A set of equations describing the chemical reactions and mixing in the wake for finite lags can similarly be written and constitutes a simple generalization of the work of Lin and Hayes.

In the far wake, where the turbulent velocities have presumably died down, the foregoing model is not expected to hold. The eddy diffusivity is probably no longer very important, and the break up of the lumps no longer occurs very efficiently. The temperature inhomogeneities then decay essentially by conduction. In fact, sufficiently far downstream all velocities will have essentially died down. Thus, the final steps of thermal equalization proceed purely by molecular diffusion. The limits of application of the two-fluid model is undoubtedly dependent on Reynolds number and body geometry, and it may not be applicable at all when the turbulent intensity is very low, corresponding to very small Reynolds numbers.

2.3 Equations Describing Quasi-One-Dimensional Model for Wake Mixing with Lag

The model of a wake consisting of homogeneously mixed old fluid and unmixed new fluid in various stages of break up can be described by a quasi-one-dimensional wake model, in which a mean-turbulent-wake boundary is assumed to be definable. The model is illustrated in Fig. 1. The fluctuations in the turbulent front are ignored. The growth of the turbulent wake is represented by the flux of outer inviscid fluid into the turbulent core through the wake boundary. The entering fluid then forms part of the unmixed portion of the core for a mean distance ζ (measured in body diameters), after which it is homogeneously mixed with the remainder of the wake by molecular effects. The actual molecular mixing of fluid entering at a particular station in the wake need not occur suddenly after a distance ζ, but may be spread over a (small) range of lags centered about ζ. This is discussed further in the following section.

The turbulent wake is then characterized by the pressure (assumed constant across the wake), and by the velocity, density, enthalpy, and chemical composition (if chemistry is included) of the homogeneously mixed and unmixed portions of the wake, together with the relative fractions of each, as a function of downstream distance. The values of the velocity, density, enthalpy, and chemical composition of the inviscid gas at the edge of the turbulent core must be assumed to be specified. The wake evolution can then be described by the one-dimensional equations of conservation of mass, momentum, and energy, by an equation describing the mixing lag, and by the equations expressing the chemical reactions or the thermodynamic characteristics of the gas (equations of state) if chemical reactions can be ignored. In the latter case, which is applicable to wakes of relatively low-speed projectiles in ballistic ranges for instance, the conservation equations specifying the quasi-one-dimensional model can be expressed as follows (z denotes downstream distance):

Conservation of Mass

$$\frac{d}{dz}(\rho_u V_u A_u) + \frac{d}{dz}(\rho_m V_m A_m) = \rho_i V_i \frac{dA}{dz} \tag{3}$$

Conservation of Momentum

$$\frac{d}{dz}(\rho_u V_u^2 A_u) + \frac{d}{dz}(\rho_m V_m^2 A_m) - A\frac{dp}{dz} = \rho_i V_i^2 \frac{dA}{dz} \tag{4}$$

Conservation of Energy

$$\frac{d}{dz}\left\{\rho_u V_u\left(h_u + \frac{V_u^2}{2}\right)A_u\right\} + \frac{d}{dz}\left\{\rho_m V_m\left(h_m + \frac{V_m^2}{2}\right)A_m\right\} = \rho_i V_i H \frac{dA}{dz} \tag{5}$$

The foregoing equations are the usual conservation equations for one-dimensional flow in a channel with mass addition at the boundary (at the same pressure as in the channel), for a flow consisting of two components. In the foregoing equa-

tions, ρ_i is the density of the inviscid gas at the edge of the turbulent core, V_i is its velocity, H is its total enthalpy, $H = [h_i + (V_i^2/2)]$, and p is the pressure. The density ρ, velocity V, and static enthalpy h, for the mixed component and the mean values of those quantities for the unmixed portions of the turbulent wake are denoted by the subscripts m and u, respectively. The areas A_m and A_u are the partial areas of hot and cold fluid, and their sum is equal to the wake area A:

$$A_m + A_u = A \tag{6}$$

Thus, it has been assumed in the foregoing formulation that the unmixed fluid can be characterized by a single average value of ρ and V. (As discussed below, it is reasonable to assume that the value of stagnation enthalpy of the unmixed component can simply be taken as the inviscid value.) This assumption can be relaxed by making substitutions of the type:

$$\rho_u V_u A_u \rightarrow \int \rho_u V_u dA_u(\rho_u, V_u) \tag{7}$$

The distribution of values of ρ_u, V_u in the unmixed portion would then have to be specified. (For instance, the distribution in the corresponding laminar flow field could be chosen.)

The hypothesis of a lag in the mixing of ingested inviscid fluid can be expressed by an equation of the form

$$(d/dz)(\rho_m V_m A_m) = \rho_i(z - \zeta) V_i(z - \zeta)[dA(z - \zeta)/dz] \tag{8}$$

where ζ is the lag distance defined previously.

Equation (8) simply expresses the assumption that fluid entering the turbulent core at station $z - \zeta$ becomes a part of the well-mixed component at station z. In addition to the conservation equations and to the lag equation, the equations of state for the mixed and unmixed components are required, as well as some additional equations or assumptions. Thus, one may assume that the pressure and the wake or channel area are given as a function of downstream distance. In that case, only one additional assumption or equation is required. An attractive choice is to assume that the total enthalpy of the unmixed component remains constant. This amounts to assuming that there is no heat transfer between the mixed and unmixed components prior to mixing, since the stagnation enthalpy is indeed constant throughout the inviscid field. The assumption is quite reasonable since diffusion and conductivity have much the same characteristic times.

Other choices that could be made might be to specify the entropy increase on mixing of the two components or to write an equation for the increase in momentum flux of the mixed component corresponding to Eq. (8), which concerns the mass flux. Still another possibility would be to assume that the unmixed fluid properties are known.

Equations (3–8) constitute a simple set of equations describing a (one-dimensional) turbulent wake with a mixing lag. In order to solve those equations, the lag ξ must be specified as a function of downstream distance. The specification of the lag is taken up in the next subsection, and a numerical solution to a simplified set of equations is presented in Sec. 3. In concluding the present section, it may be useful to note that the present model makes no identifiable assumption about the nature of the process which converts u-fluid into m-fluid. The physics of this mixing, so to speak, is partly hidden in the specification of pressure and wake growth, and partly, of course, in the assumption of a one-dimensional model. It should also be noted that the preceeding mathematical model is consistent with the phenomelogical description of wake mixing, but cannot be considered to be deduced with any rigor from the physical wake model.

2.4 Mixing Lag Distance

In Eq. (8), the assumed lag in the molecular mixing of ingested fluid into the core is represented by a lag distance ζ.

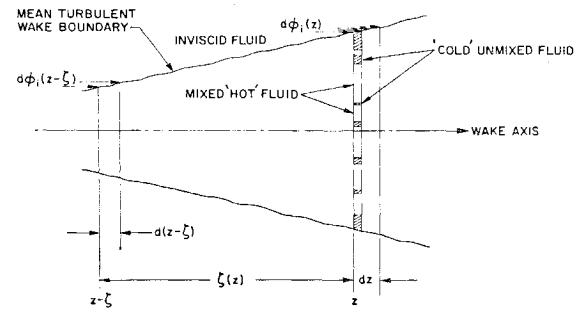

Fig. 1 Wake mixing with lag. The fluid entering the mean-turbulent boundary at $z - \zeta$ travels a distance ζ before mixing on the molecular level with the gas already present in the wake core. Thus the change in the core density and temperature over the distance dz is controlled by the fluid entering the turbulent core within $d(z - \zeta)$. The lag in mixing leads to a turbulent core composition consisting of a hot, homogeneously mixed portion, and a cold unmixed portion.

This distance is approximately equal to some lag time τ multiplied by the mean wake velocity. The lag time τ is clearly related to the time scale for dissipation of inhomogeneities in a turbulent flow. This lag is determined by the rate of molecular mixing in a turbulent flow field, and is, therefore, related to the rate of destruction of scalar inhomogeneities in such a flow field. The only such rate presently available from the theory of turbulence is that for the mixing of a passive scalar in a homogeneous steady-state field of turbulence[5, 13] and even then the rate is only known at large values of the time, when the effect of the initial distribution of scalar inhomogeneities has disappeared. In that case, the (logarithmic) rate for the dissipation of scalar inhomogeneities is the same as the dissipation rate of kinetic energy. There is, of course, no good reason to believe that the distribution of inhomogeneities in the wake mixing problem would be the same as that in such a homogeneous steady-state field of turbulence. However, it is not unreasonable to hope that the mean rate of dissipation of inhomogeneities predicted for steady-state homogeneous turbulence will not be too far off in the present problem. Within such an approximation, the time scale for dissipation§ of scalar inhomogeneities is[13]

$$\tau = \alpha \int_{1/D}^{1/\eta} \epsilon^{-1/3} k^{-5/3} dk \tag{9}$$

where k is the eddy wave number, ϵ the dissipation rate of kinetic energy, η the Kolmogorov scale, and α is a constant of order unity. D was previously defined as the initial scale of inhomogeneities and is assumed given by Eq. (2). From Eq. (9), we write approximately

$$\zeta = \alpha V \int_{1/D}^{1/\eta} \epsilon^{-1/3} k^{-5/3} dk \tag{10}$$

If the result of subsonic turbulence [that the ratio of mean flow to turbulent kinetic energy remains approximately constant throughout most of the wake (as long as the local flow Reynolds number is high)] is taken over to the present case, and a power law

$$(\Delta V/V_\infty)^2 = Kz^{-n} \tag{11}$$

is assumed for the mean-velocity decay in the wake, it can be shown that ζ may be approximately written

$$\zeta = z\{1 - \exp[-(\rho/\rho_\infty)^{2/3}/n]\} \tag{12}$$

§ Note that, in actual fact, since the wake is engulfing new fluid, inhomogeneities are being generated as well as dissipated. Thus, Eq. (9) is to be used only to determine a mean dissipation or mixing time (or distance).

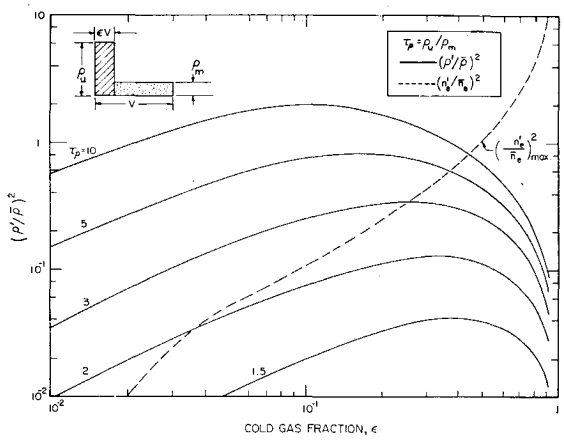

Fig. 2 Variation of mean-square mass and electron-density oscillations with the cold component fraction ϵ in the coarse-grained wake model; the ratio of cold to hot gas densities is used as a parameter.

ζ is proportional to z over regions of constant $n(\rho/\rho_\infty)$. In the subsequent calculations, ζ is approximated by linear segments. A similar result is obtained if the lag ζ is directly related to the dissipation rate of kinetic energy turbulence in the wake, by an equation of the form $1/\zeta \sim d/dz \ln u'^2$.

Clearly one cannot expect Eq. (12) to provide an exact estimate of the lag. In fact, such an exact estimate is not very meaningful, since ζ is intended to represent some mean-mixing lag, with the actual mixing of any given fluid element occurring over some range of downstream distances. Such a range in mixing distance could be expected to arise from the fact that the survival time τ of a fluid element will exhibit a range of values, from the fact that a given element will not mix entirely at one time, and finally from the fact that there is a velocity profile in the wake, leading to a range in ζ even for a fixed survival time. However, if that range is fairly small compared to the length scale over which the outer (inviscid) field properties vary appreciably, a sharp lag ζ can be used to represent mixing of a fluid element over a range of distance after it is ingested. Indeed, a more general form of Eq. (8) representing mixing over a range of downstream distance might be given as¶

$$\frac{d}{dz}(\rho_m V_m A_m) = \int_{z_0}^{z} \rho_i(z')\, V_i(z') \frac{dA(z')}{dz'} \frac{\partial \phi(z,z')}{\partial z}\, dz' \quad (13)$$

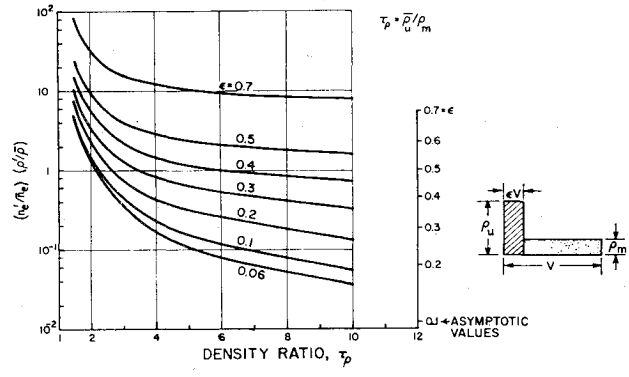

Fig. 3 Ratio of mean-square electron density to mass-density oscillations as a function of the density ratio of the cold and hot wake components in the coarse-grained wake model; the fraction ϵ of unmixed fluid is used as a parameter.

¶ The use of a lag equation summing the contributions to the mixed flow at a given station from the unmixed flux at all preceding stations was suggested by A. Nayfeh of Heliodyne Corporation, though in a somewhat different form.

where $\phi(z, z')$ is the portion of the inviscid flux per unit area entering the core at z' which has survived to station z, and therefore, $\partial\phi/\partial z$ is a rate of mixing at station z of inviscid fluid which entered the core at z'. (Obviously $\phi \equiv 0$ when $z < z'$.)

The discussion of mixing in a turbulent flow presented in the preceding sections suggests that $\partial\phi/\partial z$ is a function which peaks at some particular value of $z - z'$, namely, when the shredding effect of turbulence has reduced the scale of the inhomogeneities to the Kolmogorov scale, and is small everywhere else.

Thus, particularly in the case of slender body wakes (where the inviscid flow field does not vary much), but without much error in all cases where chemical reactions do not make the flow properties vary too rapidly with distance, we may rewrite Eq. (13) as

$$\frac{d}{dz}(\rho_m V_m A_m) \simeq \rho_i(z-\zeta) V_i(z-\zeta) \frac{dA(z-\zeta)}{d(z-\zeta)} \int_{z_0}^{z} \frac{\partial \phi i}{\partial z}\, dz' = \rho_i(z-\zeta) V_i(z-\zeta) \frac{dA(z-\zeta)}{dz} \quad (14)$$

where ζ is now the value about which $(\partial\phi i/\partial z)$ peaks. If the peak of $(\partial\phi i/\partial z)$ is assumed infinitely sharp, $[\partial\phi i/\partial(z-\zeta)] = \delta(z-\zeta)$ and $(\partial\phi i/\partial z) = [d(z-\zeta)/dz]\delta(z-\zeta)$, confirming Eq. (8). Thus, the wake mixing formulation using a sharp lag can be used to describe gradual mixing over a range small compared to the range over which the inviscid flow properties change significantly.

2.5 Mass and Electron Density in the Two-Component Wake

The mean values of mass- and electron-density variations in any transverse slice of the wake for the present two-component wake model will now be discussed. Here it is assumed, once more, that each component is characterized by a single temperature, density, electron density, and so forth. Assuming, in addition, that a volume average over a slice of the mean wake predicted by the model can be taken to represent a statistical average, the following mean values of any quantity θ can be readily obtained:

$$\theta \equiv \frac{1}{A}\int_A \theta dA = \epsilon\theta_u + (1-\epsilon)\theta_m \quad (15)$$

$$\theta'^2 \equiv \frac{1}{A}\int_A (\theta - \bar{\theta})^2 dA = \epsilon(\theta_u - \bar{\theta})^2 + (1-\epsilon)(\theta_m - \bar{\theta})^2 \quad (16)$$

where $\bar{\theta}$ is the mean value, θ'^2 the mean-square variation of θ, and ϵ represents the fraction of unmixed fluid at any station z.

Let τ_θ be the ratio

$$\tau_\theta = \begin{cases} \theta_u/\theta_m & \text{if} \quad \theta_u \geq \theta_m \\ \theta_m/\theta_u & \text{if} \quad \theta_u \leq \theta_m \end{cases} \quad (17)$$

Then from Eqs. (15–17), we may write:

$$\left(\frac{\theta'^2}{\bar{\theta}^2}\right) = \frac{\epsilon(1-\epsilon)(\tau_\theta - 1)^2}{[1+\epsilon(\tau_\theta - 1)]^2} \quad \text{if} \quad \theta_u > \theta_m \quad (18a)$$

$$\left(\frac{\theta'^2}{\bar{\theta}^2}\right) = \frac{\epsilon(1-\epsilon)(\tau_\theta - 1)^2}{[1+(1-\epsilon)(\tau_\theta - 1)]^2} \quad \text{if} \quad \theta_u < \theta_m \quad (18b)$$

Equation (18a) or (18b) expresses a necessary relationship between the mean-square deviation of θ, the ratio of its extreme values θ_u and θ_m, and the relative proportion ϵ of unmixed fluid in the wake. The only requirement for its validity is that all but a negligible portion of the wake be represented by the values θ_u and θ_m of θ.

Letting $\theta \equiv \rho$, the mass density in the wake applies, since $\rho_u > \rho_m$ and, therefore,

$$\left(\frac{\rho'}{\bar{\rho}}\right)^2 = \frac{\epsilon(1-\epsilon)(\tau_\rho - 1)^2}{[1+\epsilon(\tau_\rho - 1)]^2} \tag{19}$$

In addition, assuming the perfect gas law to be approximately valid (chemical reactions, as has been mentioned above, are neglected), the value of τ_ρ is

$$\tau_\rho = \frac{\rho_u}{\rho_m} = \frac{h_m}{h_u} = \frac{T_m}{T_u} \tag{20}$$

where T is temperature.

Letting $\theta = n_e$, the electron density in the wake yields

$$\left(\frac{n_e'^2}{\bar{n}_e^2}\right) = \frac{\epsilon(1-\epsilon)(\tau_e - 1)^2}{[1+(1-\epsilon)(\tau_e - 1)]^2} \tag{21}$$

since $(n_e)_u < (n_e)_m$. In fact, because the degree of ionization varies quite rapidly with temperature, it may be assumed that $(n_e)_u \ll (n_e)_m$, and consequently $\tau_e \gg 1$. Thus, $(n_e'^2/\bar{n}_e^2)$ may be approximated by its limiting value for infinite τ_e:

$$\left(\frac{n_e'^2}{\bar{n}_e^2}\right) \simeq \lim_{\tau_e \to \infty} \left(\frac{n_e'^2}{\bar{n}_e^2}\right) = \frac{\epsilon}{1-\epsilon} \tag{22}$$

Note that for a given value of ϵ, $(n_e'^2/\bar{n}_e^2)$ is a monotonically increasing function of τ_e, so that the limiting value is an upper limit to $(n_e'^2/\bar{n}_e^2)$ for a given value of ϵ.

The equations for $(\rho'^2/\bar{\rho}^2)$ and $(n_e'^2/\bar{n}_e^2)$ just derived indicated that these two quantities have, in general, quite different values in a wake. In particular, it may be noted that low values of ϵ tend to make the mean-mass-density oscillations large compared to the mean-electron-density oscillations and that high values of ϵ have a contrary effect. This is simply because the mass and electron densities are inversely related, so to speak, since a high mass density implies a low electron density and vice versa. The dependence of $(\rho'^2/\bar{\rho}^2)$ and $(n_e'^2/\bar{n}_e^2)$ on ϵ for various τ_ρ is depicted in Fig. 2. Similarly the ratio of mean-square electron to mass-density oscillations in the wake as a function of the temperature or density ratio τ_ρ is shown in Fig. 3 for various values of ϵ.

So far, the discussion of mass- and electron-density fluctuations has dealt with their dependence on the fraction ϵ of unmixed fluid in the turbulent wake core, and on the ratio of temperatures τ_ρ of the relatively cold unmixed gas and of the hotter homogeneously mixed gas. The actual values of those quantities and their variation with downstream distance have not been specified. In fact, a determination of those quantities for the present wake mixing model requires the solution of the set of equations (3–8) or some other set of equations which provides a simple mathematical model of the mixing. Such a solution is presented and discussed in the following section, for a very simple mathematical model.

3. Sample Numerical Calculations

In the present section, a numerical solution of a simplified set of mixing equations is presented and discussed. It should be stressed at the outset that the model is crude, and that, in particular, it does not account very well for what might be termed dynamic effects, such as drag between the mixed and unmixed components, the effects on pressure of mixing of the two components, and so forth. It is primarily what its name implies, i.e., a mixing model, and is intended for application to reacting wakes. However, the low-speed nonreacting case is of interest, particularly in comparing its predictions with experimental results. The calculations were made to correspond to the case of a 1-cm-diam body traveling at some 10,000 fps.

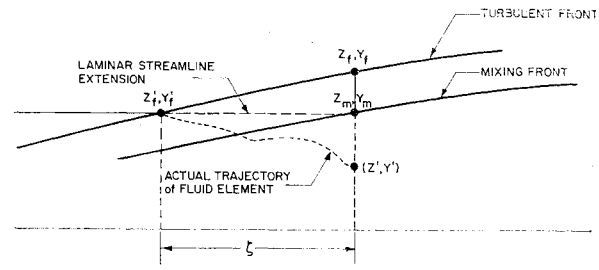

Fig. 4 Effective mixing boundary; a fluid element entering the turbulent core at (z_f', y_f') is assumed to survive a distance ζ and thus effectively mix at (z_m, y_m), although its actual location may be (z', y').

3.1 Basis for Numerical Calculations

The equations used in the numerical computations presented below are the standard equations for one-dimensional flow in a channel, with mass injection at the boundary. These equations result from the mixing with lag equations if the unmixed component properties are assumed to remain the same as in a laminar flow, prior to mixing, so that the flow in the channel describes only the mixed component. The lag is introduced by specification of an effective mixing boundary, which is described below and does not coincide with the (mean) turbulent boundary.

The relation of the effective mixing boundary to the mean-turbulent boundary is depicted in Fig. 4. Inviscid fluid entering the turbulent boundary at (z_f', y_f') survives molecular mixing for a distance ζ, so that it would have reached the point (x_m, y_m) if it had continued along its laminar trajectory. Since the random convection by the turbulent velocity field in the wake is assumed not to affect the fluid, it is effectively mixed at (z_m, y_m). Thus, the mixing boundary can be constructed from the mean-wake boundary by simply following fluid elements on an extension of the laminar streamlines from their point of crossing of the turbulent boundary to a point ζ downstream. The amount of unmixed fluid is then that contained between y_f and y_m.

The mean-wake boundary used in the numerical computations was obtained from a $\frac{1}{3}$ power law fitted to experiment. The lag function was approximated by a linear dependence of ζ on downstream distance over several wake segments, with the coefficients chosen to roughly match Eq. (12). The value of n in Eq. (12) over the various segments was estimated by reference to measured data on velocity decay in wakes,[15] and $(\bar{\rho}/\rho_\infty)$ was estimated from a theoretical calculation by Hromas.[14] From such estimates, the lag was assumed to rise from 0.05 z, for z less than 50 body diameters, to 0.1 z between 50 and 150 diameters, and to increase to 0.5 z for z greater than 300 diameters, with additive constants used to make ζ continuous. Beyond 500 diameters, ζ was approximated to reach a constant value corresponding to a lag time of approximately 2 msec, which is approximately the diffusion time for molecular diffusion from a 1-cm-diam sphere at a temperature of approximately 500°K and 0.06 atmospheric pressure.

The specification of the inviscid flow outside the turbulent core was obtained from a hemisphere cylinder solution also supplied by Hromas.[14] The freestream velocity in the numerical case used was about 9600 fps, and the freestream density used was $(\rho_\infty/\rho_0) \sim 6 \times 10^{-2}$ corresponding to an altitude of 60,000 ft. An equilibrium gas was assumed in the numerical computations.

3.2 Results and Discussion

The results obtained by solving the one-dimensional channel flow with an effective mixing boundary are shown in Figs. 5–7. Figure 5 shows the variation of the density

ρ_m along the wake as well as the mean density $\bar{\rho}_u$ of the unmixed component. It can be seen that $\bar{\rho}_u$ becomes substantially greater than ρ_m as we proceed downstream. This leads to a relatively large ratio $\tau_0 = \rho_u/\rho_m$ and the possibility of large relative fluctuations, as discussed in subsection 2.4. The ratio τ_0 is plotted in Fig. 6 along with the unmixed volume fraction ϵ.

Finally, the relative root-mean-square gas density fluctuation $(\rho'/\bar{\rho})$ is plotted in Fig. 7. The gas density fluctuation is obtained from the equation

$$\frac{\rho}{\bar{\rho}} = \frac{\epsilon\,(\rho_u^2 - \bar{\rho}_u^2) + \epsilon\,(1-\epsilon)(\bar{\rho}_u - \rho_m)^2}{\bar{\rho}} \qquad (23)$$

which accounts for fluctuations arising from inhomogeneity of the unmixed gas. Also shown in Fig. 7 are some experimental measurements of gas density fluctuations obtained by schlieren techniques for wakes of pellets traveling at approximately 8500 to 9000 fps.[6] The measurements are not definitive, in the sense that their absolute magnitude is still uncertain at this time.[6] However, measurements subsequent to those for which the data is shown indicate[6] the same general trend in the data, but with a significantly lower maximum occurring at three or four hundred body diameters downstream. (The maximum is closer to the predicted value than previous measurements indicated.) It should be stressed, however, that, to the authors' knowledge, the absolute magnitude of the fluctuations measured by the schlieren method is not yet firmly established.

4. Concluding Remarks

The present paper has been based upon a characterization of wake mixing in terms of propagation of turbulence by small scale eddies, random convection of fluid elements within the turbulent core by large scale eddies, and dissipation of inhomogeneities by the combined action of distortion of fluid elements by turbulent motions and molecular diffusion. Such a characterization is completely consistent with the theory of (low-speed) turbulence as it exists today. It leads to a two-component wake model, in which one component consists of fluid which is relatively homogeneous and well mixed, whereas the other consists of fluid newly entrained into the turbulent core, and, therefore, retaining its unmixed properties. In the preceding model, the time delay in thorough (molecular) mixing of the latter component was represented by a lag time or lag distance. A set of one-dimensional equations derived for such a model was developed, and the solution to a simplified version of those equations for a relatively low-speed wake provided predictions of density fluctuations in the wake in good agreement with available experimental data.

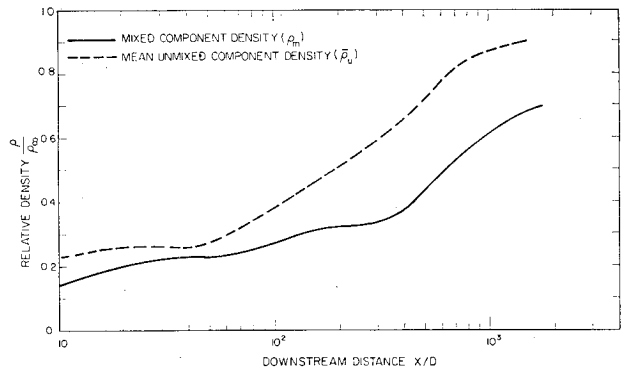

Fig. 5 Variation of mixed fluid density ρ_m and mean-unmixed fluid density ρ_u with downstream distance in mixing with lag model.

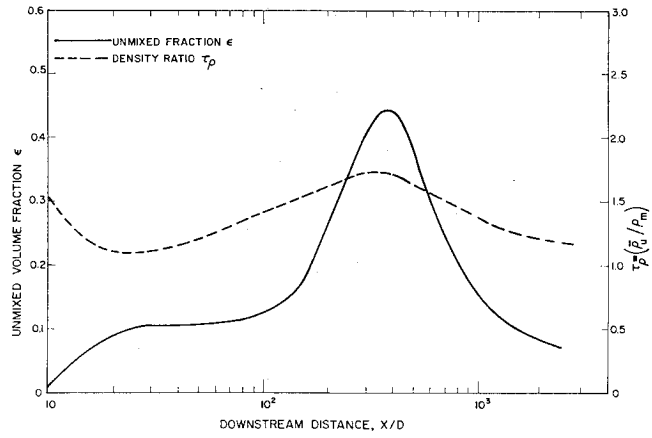

Fig. 6 Variation of ratio of mixed fluid density to mean-unmixed fluid density and of unmixed volume fraction ϵ with downstream distance.

The model developed previously is intended for use in predicting the properties of turbulent wakes of high-speed reentry vehicles. For such wakes, the chemical reactions in the wake are of primary importance, and the model requires extension and further refinement in order to provide reliable predictions for reacting wakes of high-speed bodies, though it can be used for rough calculations by including reactions in the conservation equations for each species of importance. In any event, the ability of the model to predict some of the features of wakes of low-speed bodies is an encouraging indication of the validity of the concept of a two-component wake.

Finally, the validity of the model could be more critically tested if measurements of both gas- and electron-density fluctuations in the wake were available, as the model makes definite predictions concerning the relation between those fluctuations.

Appendix

In this Appendix, a rough estimate of the magnitude of density oscillations which can be generated in a turbulent wake by dissipation and compressibility effects is provided. The order of magnitude of the density fluctuations that could be generated in a turbulent wake may be estimated as follows. Assume that the source of the fluctuations is dissipation or pV work and is generated by the fluctuating velocities rather than by convection of cold high-density gas into hot low-density gas. Since there is no way to support large pressure fluctuations in a free flow like a wake, beyond a few downstream diameters (particularly for freestream velocities of 10^4 fps or so), we assume that the fluctuations occur at constant pressure. For the case of fluctuations generated by viscous dissipation, it may be assumed that, at most, all the random kinetic energy per unit mass in a given unit mass of fluid will be converted into heat in generating a density change.**

The kinetic energy E_{turb} per unit mass in the random velocity field is approximately

$$E_{\text{turb}} = \tfrac{3}{2}\,u'^2 = \tfrac{3}{2}\,(u'^2/\Delta V^2)(\Delta V/V_\infty)^2\,V_\infty^2 \qquad (A1)$$

where u' is the root-mean-square turbulent velocity, ΔV is the wake velocity defect, and V_∞ is the freestream velocity.

The ratio $(u'^2/\Delta V^2)$ can be estimated to be roughly one-tenth beyond a few tens of body diameter. The approximate relation (see Sec. 2.1)

$$(\Delta V/V_\infty) \sim (\tfrac{1}{8}y_f^2)(\rho_\infty/\rho) \qquad (A2)$$

may be used for the wake velocity decay.

** This is an upper limit by any standard.

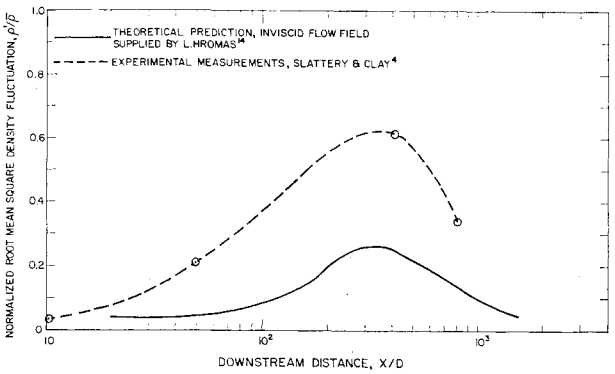

Fig. 7 Normalized mass-density fluctuations along wake according to mixing with lag model; the experimental points shown have been reduced by a factor of $2^{1/2}$ to correct for an error in data reduction, at the suggestion of the experimenters.[4,6]

On the other hand, assuming that all the kinetic energy of turbulence is converted to heat, so that $c_p\Delta T \sim E_{\text{turb}}$, the resulting density change is

$$\frac{\Delta\rho}{\rho} = -\frac{\Delta T}{T} = -\frac{E_{\text{turb}}}{c_p T} = -\frac{3}{2}\left(\frac{u'}{\Delta V}\right)^2\left(\frac{\Delta V}{V_\infty}\right)^2\left(\frac{V_\infty^2}{h}\right) \quad \text{(A3)}$$

where h is the local static enthalpy.

The ratio (V_∞^2/h) can be stimulated by writing it in the form

$$\frac{V_\infty^2}{h} = \frac{V_\infty^2}{h_\infty + \frac{1}{2}(V_\infty^2 - V_w^2)} \approx \frac{2V_\infty}{\Delta V} \quad \text{(A4)}$$

Finally, substituting Eqs. (A1, A2, and A4) into (A3), and assuming $y_f \sim \frac{1}{2}z^{1/3}$, the density fluctuation can be estimated as

$$|\Delta\rho/\rho| \sim 0.15\, z^{-2/3}(\rho_\infty/\bar{\rho}) \quad \text{(A5)}$$

where $\bar{\rho}$ is the mean wake density. From the preceding equation, it is clear that, at downstream distances of the order of one to several hundred body diameters, the density fluctuations due to dissipation can be expected to be of the order of 1% or less. The same estimates can be shown to result if it is assumed that the density fluctuations are generated by conversion of turbulence energy into pV work. Thus, it is safe to assume that, if fluctuations in density of the order of several tens percent do exist in the wake, as some evidence indicates, they must arise from a random mixing effect.

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